STABILITY OF PERIODIC WAVES OF FINITE AMPLITUDE ON THE SURFACE OF A DEEP FLUID

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ABSTRACT: We study the stability of steady nonlinear waves on the surface of an infinitely deep fluid [1, 2]. In section 1, the equations of hydrodynamics for an ideal fluid with a free surface are transformed to canonical variables: the shape of the surface $\eta(\mathbf{r}, t)$ and the hydrodynamic potential $\Psi(\mathbf{r}, t)$ at the surface are expressed in terms of these variables. By introducing canonical variables, we can consider the problem of the stability of surface waves as part of the more general problem of nonlinear waves in media with dispersion [3, 4]. The results of the rest of the paper are also easily applicable to the general case.

In section 2, using a method similar to van der Pohl's method, we obtain simplified equations describing nonlinear waves in the small amplitude approximation. These equations are particularly simple if we assume that the wave packet is narrow. The equations have an exact solution which approximates a periodic wave of finite amplitude.

In section 3 we investigate the instability of periodic waves of finite amplitude. Instabilities of two types are found. The first type of instability is destructive instability, similar to the destructive instability of waves in a plasma [5, 6]. In this type of instability, a pair of waves is simultaneously excited, the sum of the frequencies of which is a multiple of the frequency of the original wave. The most rapid destructive instability occurs for capillary waves and the slowest for gravitational waves. The second type of instability is the negative-pressure type, which arises because of the dependence of the nonlinear wave velocity on the amplitude; this results in an unbounded increase in the percentage modulation of the wave. This type of instability occurs for nonlinear waves through any media in which the sign of the second derivative in the dispersion law with respect to the wave number ($d^2\omega/dk^2$) is different from the sign of the frequency shift due to the nonlinearity.

As announced by A. N. Litvak and V. I. Talanov [7], this type of instability was independently observed for nonlinear electromagnetic waves.

1. Canonical variables. We consider the potential flow of an ideal fluid of infinite depth in a homogeneous gravity field. We choose the coordinates so that the undisturbed surface of the fluid coincides with the xy-plane. The z-axis is directed away from the surface. In what follows, all vectors are two-dimensional vectors in the xy-plane.

Let $\eta(\mathbf{r}, t)$ be the shape of the surface of the fluid and let $\Phi(\mathbf{r}, z, t)$ be the hydrodynamic potential. The fluid flow is described by Laplace's equation,

$$\Delta\Phi + \frac{\partial^2\Phi}{\partial z^2} = 0, \tag{1.1}$$

with two conditions at the surface,

$$\frac{\partial\eta}{\partial t} = \sqrt{1+\nabla\eta^2}\,\frac{\partial\Phi}{\partial n}\Big|_{z=\eta} = \frac{\partial\Phi}{\partial z} - \nabla\eta\,\nabla\Phi\,|_{z=\eta}, \tag{1.2}$$

$$\frac{\partial\Phi}{\partial t} + g\eta = -\frac{1}{2}(\nabla\Phi)^2|_{z=\eta} - \frac{1}{2}\left(\frac{\partial\Phi}{\partial z}\right)^2\Big|_{z=\eta} + \alpha\nabla\frac{\nabla\eta}{\sqrt{1+\nabla\eta^2}},$$

and a condition at infinity,

$$\Phi \to 0 \quad \text{as} \quad z \to -\infty.$$

Here g is the acceleration due to gravity and α is the coefficient of surface tension.

Equations (1.1)–(1.3) conserve the total energy of the fluid,

$$E = \frac{1}{2}\int d\mathbf{r}\int_{-\infty}^{\eta}\left[(\nabla\Phi)^2 + \left(\frac{\partial\Phi}{\partial z}\right)^2\right]dz +$$
$$+\frac{1}{2}g\int\eta^2 d\mathbf{r} + \alpha\int\left(\sqrt{1+\nabla\eta^2} - 1\right)d\mathbf{r}. \tag{1.4}$$

The first term in this expression is the kinetic energy and the second and third terms are the potential energy in the field of gravity and the potential energy due to surface forces. We introduce the quantity $\Psi(\mathbf{r}, t) = \Phi(z, \mathbf{r}, t)|_{z=\eta}$. Specifying the quantities η and Ψ fully defines the fluid flow since the boundary-value problem for Laplace's equation has a unique solution. Using the equation

$$\frac{\partial\Psi}{\partial t} = \frac{\partial\Phi}{\partial t} + \frac{\partial\eta}{\partial t}\frac{\partial\Phi}{\partial z}\Big|_{z=\eta},$$

we obtain

$$\frac{\partial\Psi}{\partial t} + g\eta - \alpha\nabla\frac{\nabla\eta}{\sqrt{1+\nabla\eta^2}} = -$$
$$-\frac{1}{2}(\nabla\Phi)^2 + \frac{1}{2}\left(\frac{\partial\Phi}{\partial z}\right)^2 - \frac{\partial\Phi}{\partial z}(\nabla\Phi\,\nabla\eta)\,|_{z=\eta}. \tag{1.5}$$

Equations (1.2) and (1.5), together with Laplace's equation, are equivalent to Eqs. (1.1)–(1.3). We can prove that Eqs. (1.1) and (1.5) can be put in the form

$$\frac{\partial\eta}{\partial t} = \frac{\delta E}{\delta\Psi}, \quad \frac{\partial\Psi}{\partial t} = -\frac{\delta E}{\delta\eta}. \tag{1.6}$$

Here E is the energy; the symbols $\delta E/\delta\eta$ and $\delta E/\delta\Psi$ denote the variational derivative.

Consider first the variation of Ψ. Obviously, the variation of the potential energy is zero. We transform the kinetic energy by means of Green's formula:

$$E^* = \frac{1}{2}\int d\mathbf{r}\int_{-\infty}^{\eta} dz\left[(\nabla\Phi)^2 + \left(\frac{\partial\Phi}{\partial z}\right)^2\right] = \frac{1}{2}\int_s \Psi\frac{\partial\Phi}{\partial n}ds =$$
$$= \frac{1}{2}\int_s \Psi\frac{\partial\Phi}{\partial n}\sqrt{1+\nabla\eta^2}d\mathbf{r}.$$

Here ds is a differential surface element. The normal derivative $\partial\Phi/\partial n$ is linked with Ψ by the Green's function for the boundary-value problem of Laplace's equation:

$$\frac{\partial\Phi(s)}{\partial n} = \int G(s, s_1)\,\Psi(s_1)\,ds_1. \tag{1.7}$$

Here s and s_1 are points on the surface. The function G is symmetric; i.e., $G(s, s_1) = G(s_1, s)$.

The variation of the kinetic energy has two terms:

$$\delta E^* = \frac{1}{2}\int_s\left[\delta\Psi(s)\frac{\partial\Phi(s)}{\partial n} + \Psi(s)\frac{\partial}{\partial n}\delta\Phi(s)\right]ds.$$

From (1.7) and the symmetry of the Green's function, we see that both terms are the same:

$$\delta E^* = \int_s \delta\Psi(s)\frac{\partial\Phi(s)}{\partial n}ds = \int_s \delta\Psi(\mathbf{r})\frac{\partial\Phi}{\partial n}\sqrt{1+\nabla\eta^2}\,d\mathbf{r}.$$

From this we obtain (1.2) instantaneously.

Consider now variation in η (this simple proof is due to R. M. Garipov).

Variation of the potential energy at once gives the terms on the left-hand side of (2.5). Variation of the kinetic energy gives

$$\delta E^* = \frac{1}{2}\int\left[(\nabla\Phi)^2 + \left(\frac{\partial\Phi}{\partial z}\right)^2\right]\delta\eta(\mathbf{r})\,d\mathbf{r} +$$
$$+\int d\mathbf{r}\int_{-\infty}^{\eta}\left[(\nabla\Phi, \nabla\delta\Phi) + \left(\frac{\partial\Phi}{\partial z}, \frac{\partial}{\partial z}\delta\Phi\right)\right]dz$$

$$\left(\delta\Phi = \frac{\partial\Phi}{\partial z}\delta\eta\right).$$

Here $\delta\Phi$ is the variation in the potential due to a change in the boundary. Since Φ satisfies Laplace's equation, we can apply the Green's theorem to the second integral:

$$\int d\mathbf{r}\int_{-\infty}^{\eta} dz\left[(\nabla\Phi,\ \nabla\delta\Phi)+\left(\frac{\partial\Phi}{\partial z},\ \frac{\partial}{\partial z}\delta\Phi\right)\right]=\int\frac{\partial\Phi}{\partial n}\delta\Phi\, ds=$$

$$=\int\left(-\frac{\partial\Phi}{\partial z}+\nabla\eta\,\nabla\Phi\right)\frac{\partial\Phi}{\partial z}\bigg|_{z=\eta}\delta\eta(\mathbf{r})\, d\mathbf{r}.$$

Finally we have

$$\delta E^{*}=\int\left[\frac{1}{2}(\nabla\Phi)^{2}-\frac{1}{2}\left(\frac{\partial\Phi}{\partial z}\right)^{2}+(\nabla\eta\,\nabla\Phi)\frac{\partial\Phi}{\partial z}\right]_{z=\eta}\delta\eta(\mathbf{r})\, d\mathbf{r}.$$

Hence we obtain (1.5).

Thus, Eqs. (1.2) and (1.5) are Hamilton's equations and Ψ and η are canonical variables, Ψ being a generalized coordinate and η a generalized momentum. The energy E of the fluid is the Hamiltonian.

To close Eqs. (1.1) and (1.5) we have to solve the boundary-value problem for Laplace's equation. We find the solution of this problem in the form of a series in powers of η. If we apply a Fourier transformation to the variables x and y,

$$\eta(\mathbf{k})=\frac{1}{2\pi}\int\eta(\mathbf{r})\, e^{-i(\mathbf{kr})}d\mathbf{r},\qquad \Psi(\mathbf{k})=\frac{1}{2\pi}\int\Psi(\mathbf{r})\, e^{-i(\mathbf{kr})}d\mathbf{r},$$

we obtain the series in a more convenient form.

Omitting the details, we immediately give the result of the expansion (up to second-order terms):

$$\Phi(\mathbf{k},\ z)=e^{|\mathbf{k}|z}\Big\{\Psi(\mathbf{k})+\int\Psi(\mathbf{k}_1)\,\eta(\mathbf{k}_2)\,|\mathbf{k}_1|\,\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\, d\mathbf{k}_1 d\mathbf{k}_2-$$

$$-\frac{1}{2}\iint\Big[(|\mathbf{k}-\mathbf{k}_3|+|\mathbf{k}-\mathbf{k}_2|-|\mathbf{k}|)\times$$

$$\times|\mathbf{k}_1|\,\Psi(\mathbf{k}_1)\,\eta(\mathbf{k}_2)\,\eta(\mathbf{k}_3)\,\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\Big]d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3\Big\}. \qquad (1.8)$$

Here δ denotes the delta function.

If we linearize (1.2) and (1.5) and consider only the first term in (1.8), we obtain the theory of small oscillations for the surface of a fluid, which describes the propagation of waves with dispersion law

$$\omega(\mathbf{k})=\sqrt{g|\mathbf{k}|+\alpha|\mathbf{k}|^{3}}.$$

We now complete the transformation to the complex variable $a(\mathbf{k})$ via the equations

$$\eta(\mathbf{r},\ t)=\frac{1}{2\pi\sqrt{2}}\int\frac{|\mathbf{k}|^{1/2}}{\omega^{1/2}(\mathbf{k})}\left[a(\mathbf{k})\, e^{i(\mathbf{kr})}+a^{*}(\mathbf{k})\, e^{-i(\mathbf{kr})}\right]d\mathbf{k},$$

$$\Psi(\mathbf{r},\ t)=-\frac{i}{2\pi\sqrt{2}}\int\frac{\omega^{1/2}(\mathbf{k})}{|\mathbf{k}|^{1/2}}\left[a(\mathbf{k})\, e^{i(\mathbf{kr})}-a^{*}(\mathbf{k})\, e^{-i(\mathbf{kr})}\right]d\mathbf{k}. \qquad (1.9)$$

Then

$$\eta(\mathbf{k})=\frac{1}{\sqrt{2}}\frac{|\mathbf{k}|^{1/2}}{\omega^{1/2}(\mathbf{k})}\left[a(\mathbf{k})+a^{*}(-\mathbf{k})\right];$$

$$\Psi(\mathbf{k})=-\frac{i}{\sqrt{2}}\frac{\omega^{1/2}(\mathbf{k})}{|\mathbf{k}|^{1/2}}\left[a(\mathbf{k})-a^{*}(-\mathbf{k})\right]. \qquad (1.10)$$

Transformation (1.9) can be considered a canonical transformation (with complex coefficients) to the variables $ia^{*}(\mathbf{k})$ and $a(\mathbf{k})$; Hamilton's equation (1.6) becomes the single equation

$$\frac{\partial a(\mathbf{k})}{\partial t}=-i\frac{\delta E}{\delta a^{*}(\mathbf{k})}.$$

Using (1.4), (1.8), and (1.10), we can express the energy in the form of a series in powers of $a(\mathbf{k})$ and $a^{*}(\mathbf{k})$:

$$E=\int\omega(\mathbf{k})\, a(\mathbf{k})\, a^{*}(\mathbf{k})\, d\mathbf{k}+\iint\Big[V(-\mathbf{k},\ \mathbf{k}_1,\ \mathbf{k}_2)\times$$

$$\times\left[a^{*}(\mathbf{k})\, a(\mathbf{k}_1)\, a(\mathbf{k}_2)+a(\mathbf{k})\, a^{*}(\mathbf{k}_1)\, a^{*}(\mathbf{k}_2)\right]\times$$

$$\times\ \delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\Big]d\mathbf{k}\, d\mathbf{k}_1\, d\mathbf{k}_2+$$

$$+\frac{1}{3}\iint\Big[V(\mathbf{k},\ \mathbf{k}_1,\ \mathbf{k}_2)\left[a^{*}(\mathbf{k})\, a^{*}(\mathbf{k}_1)\, a^{*}(\mathbf{k}_2)+a(\mathbf{k})\, a(\mathbf{k}_1)\, a(\mathbf{k}_2)\right]\times$$

$$\times\ \delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\Big]d\mathbf{k}\, d\mathbf{k}_1\, d\mathbf{k}_2+$$

$$+\frac{1}{2}\iint\Big[W(\mathbf{k},\ \mathbf{k}_1,\ \mathbf{k}_2,\ \mathbf{k}_3)\, a^{*}(\mathbf{k})\, a^{*}(\mathbf{k}_1)\, a(\mathbf{k}_2)\, a(\mathbf{k}_3)\times$$

$$\times\ \delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\Big]d\mathbf{k}\, d\mathbf{k}_1\, d\mathbf{k}_2\, d\mathbf{k}_3, \qquad (1.11)$$

where

$$V(\mathbf{k},\ \mathbf{k}_1,\ \mathbf{k}_2)=$$

$$=\frac{1}{8\pi\sqrt{2}}\Big\{\left[(\mathbf{kk}_1)+|\mathbf{k}||\mathbf{k}_1|\right]\left(\frac{\omega(\mathbf{k})\,\omega(\mathbf{k}_1)}{\omega(\mathbf{k}_2)}\right)^{1/2}\left[(\mathbf{kk}_2)+|\mathbf{k}||\mathbf{k}_2|\right]\times$$

$$\times\left(\frac{\omega(\mathbf{k})\,\omega(\mathbf{k}_2)}{\omega(\mathbf{k}_1)}\right)^{1/2}\left(\frac{|\mathbf{k}_1|}{|\mathbf{k}||\mathbf{k}_2|}\right)^{1/2}+$$

$$+\left[(\mathbf{k}_1\mathbf{k}_2)+|\mathbf{k}_1||\mathbf{k}_2|\right]\left(\frac{\omega(\mathbf{k}_1)\,\omega(\mathbf{k}_2)}{\omega(\mathbf{k})}\right)^{1/2}\left(\frac{|\mathbf{k}|}{|\mathbf{k}_1||\mathbf{k}_2|}\right)^{1/2}\Big\}; \qquad (1.12)$$

$$W(\mathbf{k},\ \mathbf{k}_1,\ \mathbf{k}_2,\ \mathbf{k}_3)=$$

$$=-\frac{3\alpha}{32\pi^{2}}\frac{(|\mathbf{k}||\mathbf{k}_1||\mathbf{k}_2||\mathbf{k}_3|)^{1/2}}{[\omega(\mathbf{k})\,\omega(\mathbf{k}_1)\,\omega(\mathbf{k}_2)\,\omega(\mathbf{k}_3)]^{1/2}}(\mathbf{kk}_1)(\mathbf{k}_2\mathbf{k}_3)+$$

$$+\frac{1}{16(2\pi)^{2}}(|\mathbf{k}||\mathbf{k}_1||\mathbf{k}_2||\mathbf{k}_3|)^{1/2}\times$$

$$\times\Big\{\left[\frac{\omega(\mathbf{k})\,\omega(\mathbf{k}_1)}{\omega(\mathbf{k}_2)\,\omega(\mathbf{k}_3)}\right]^{1/2}(2|\mathbf{k}|+2|\mathbf{k}_1|-|\mathbf{k}-\mathbf{k}_2|-$$

$$-|\mathbf{k}-\mathbf{k}_3|-|\mathbf{k}_1-\mathbf{k}_2|-|\mathbf{k}_1-\mathbf{k}_3|)+$$

$$+\left[\frac{\omega(\mathbf{k}_2)\,\omega(\mathbf{k}_3)}{\omega(\mathbf{k})\,\omega(\mathbf{k}_1)}\right]^{1/2}(2|\mathbf{k}_2|+2|\mathbf{k}_3|-$$

$$-|\mathbf{k}-\mathbf{k}_2|-|\mathbf{k}-\mathbf{k}_3|-|\mathbf{k}_1-\mathbf{k}_2|-|\mathbf{k}_1-\mathbf{k}_3|)-$$

$$-\left[\frac{\omega(\mathbf{k}_1)\,\omega(\mathbf{k}_2)}{\omega(\mathbf{k})\,\omega(\mathbf{k}_3)}\right]^{1/2}(2|\mathbf{k}_1|+2|\mathbf{k}_2|-$$

$$-|\mathbf{k}_2+\mathbf{k}_3|-|\mathbf{k}-\mathbf{k}_2|-|\mathbf{k}-\mathbf{k}_1|-|\mathbf{k}_1+\mathbf{k}_3|)-$$

$$-\left[\frac{\omega(\mathbf{k})\,\psi(\mathbf{k}_2)}{\omega(\mathbf{k}_1)\,\omega(\mathbf{k}_3)}\right]^{1/2}(2|\mathbf{k}_1|+2|\mathbf{k}_2|-$$

$$-|\mathbf{k}_2+\mathbf{k}_3|-|\mathbf{k}_1-\mathbf{k}_2|-|\mathbf{k}-\mathbf{k}_1|-|\mathbf{k}+\mathbf{k}_3|-$$

$$-\left[\frac{\omega(\mathbf{k})\,\omega(\mathbf{k}_3)}{\omega(\mathbf{k}_1)\,\omega(\mathbf{k}_2)}\right]^{1/2}(-|\mathbf{k}_2+\mathbf{k}_3|+$$

$$+2|\mathbf{k}|+2|\mathbf{k}_3|-|\mathbf{k}-\mathbf{k}_1|-|\mathbf{k}-\mathbf{k}_3|-|\mathbf{k}+\mathbf{k}_2|)-$$

$$-\left[\frac{\omega(\mathbf{k}_1)\,\omega(\mathbf{k}_3)}{\omega(\mathbf{k})\,\omega(\mathbf{k}_2)}\right]^{1/2}(2|\mathbf{k}_1|+2|\mathbf{k}_3|-$$

$$-|\mathbf{k}_2+\mathbf{k}_3|-|\mathbf{k}-\mathbf{k}_3|-|\mathbf{k}-\mathbf{k}_s|-|\mathbf{k}_1+\mathbf{k}_2|)\Big\}. \qquad (1.13)$$

There are other fourth-order terms in a, proportional to products of the form $a^{*}aaa$ and $aaaa$ and terms conjugate to them. These are ignored, since, as will be shown in section 3, their contribution is small.

We note that the functions V and W obey the following equations:

$$V(\mathbf{k},\ \mathbf{k}_1,\ \mathbf{k}_2)=V(\mathbf{k},\ \mathbf{k}_2,\ \mathbf{k}_1)=V(\mathbf{k}_2,\ \mathbf{k}_1,\ \mathbf{k}),$$

$$V(-\mathbf{k},\ -\mathbf{k}_1,\ -\mathbf{k}_2)=V(\mathbf{k},\ \mathbf{k}_1,\ \mathbf{k}_2),$$

$$W(\mathbf{k},\ \mathbf{k}_1,\ \mathbf{k}_2,\ \mathbf{k}_3)=W(\mathbf{k}_1,\ \mathbf{k},\ \mathbf{k}_2,\ \mathbf{k}_3)= \qquad (1.14)$$

$$=W(\mathbf{k},\ \mathbf{k}_1,\ \mathbf{k}_3,\ \mathbf{k}_2)=W(\mathbf{k}_2,\ \mathbf{k}_3,\ \mathbf{k},\ \mathbf{k}_1),$$

$$W(-\mathbf{k},\ -\mathbf{k}_1,\ -\mathbf{k}_2,\ \mathbf{k}_3)=W(\mathbf{k},\ \mathbf{k}_1,\ \mathbf{k}_2,\ \mathbf{k}_3).$$

The equation for $a(\mathbf{k})$ has the form

$$\frac{\partial a(\mathbf{k})}{\partial t}+i\omega(\mathbf{k})\, a(\mathbf{k})=$$

$$=-i\int\{V(-\mathbf{k},\ \mathbf{k}_1,\ \mathbf{k}_2)\, a(\mathbf{k}_1)\, a(\mathbf{k}_2)\,\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)+$$

$$+2V(-\mathbf{k}_1, \mathbf{k}, \mathbf{k}_2)\, a^*(\mathbf{k}_2)\, a(\mathbf{k}_1)\, \delta(\mathbf{k}-\mathbf{k}_1+\mathbf{k}_2)+$$
$$+V(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)\, a^*(\mathbf{k}_1)\, a^*(\mathbf{k}_2)\times$$
$$\times\,\delta(\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2)\}\, d\mathbf{k}_1 d\mathbf{k}_2 -$$
$$-\,i\int W(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)\, \delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\times$$
$$\times\, a^*(\mathbf{k}_1)\, a(\mathbf{k}_2)\, a(\mathbf{k}_3)\, d\mathbf{k}_1\, d\mathbf{k}_2\, d\mathbf{k}_3\,. \qquad (1.15)$$

We see from (1.15) that the variables $a(\mathbf{k})$ are the normal variables in the problem of small oscillations.

2. Simplified equations. Equation (1.15) is an approximation and is valid for small nonlinearities, roughly speaking, for $a/\lambda \ll 1$, where a is the characteristic amplitude of the wave and λ is a characteristic wavelength. In this approximation, we can make a considerable simplification in Eq. (1.12). To do this we write $a(\mathbf{k})$ as

$$a(\mathbf{k}) = [A(\mathbf{k}, t) + f(\mathbf{k}, t)]\exp[-i\omega(\mathbf{k})\, t]. \qquad (2.1)$$

We assume that A(k, t) changes slowly in comparison with f, where $f \ll A$. We substitute $a(\mathbf{k})$, in the form of (2.1), into the equation for f and the one for A. In the equation for f we retain only terms which are quadratic in A. Assuming A constant as f varies, we integrate this equation with respect to time. This yields

$$f=-\int\Big\{V(-\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)\, \frac{\exp it\,[\omega(\mathbf{k})-\omega(\mathbf{k}_1)-\omega(\mathbf{k}_2)]}{\omega(\mathbf{k})-\omega(\mathbf{k}_1)-\omega(\mathbf{k}_2)}\times$$
$$\times\,\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\, A(\mathbf{k}_1)\, A(\mathbf{k}_2)+$$
$$+2V(-\mathbf{k}_1, \mathbf{k}, \mathbf{k}_2)\, \frac{\exp it\,[\omega(\mathbf{k})+\omega(\mathbf{k}_1)-\omega(\mathbf{k}_2)]}{\omega(\mathbf{k})+\omega(\mathbf{k}_1)-\omega(\mathbf{k}_2)}\times$$
$$\times\,\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2)\, A^*(\mathbf{k}_1)\, A(\mathbf{k}_2)+$$
$$+V(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)\, \frac{\exp it\,[\omega(\mathbf{k})+\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2)]}{\omega(\mathbf{k})+\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2)}\times$$
$$\times\,\delta(\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2)\, A^*(\mathbf{k}_1)\, A^*(\mathbf{k}_2)\Big\}\, d\mathbf{k}_1 d\mathbf{k}_2\,. \qquad (2.2)$$

In the equation for A we retain only those terms proportional to Af which contain the most slowly varying exponents. Obviously, all the slowly varying exponents are contained in those terms proportional to A^*AA. Gathering all these terms together, we obtain

$$\frac{\partial A(\mathbf{k})}{\partial t}=-i\int T(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)\, \delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\times$$
$$\times\,\exp it\,[\omega(\mathbf{k})+\omega(\mathbf{k}_1)-\omega(\mathbf{k}_2)-$$
$$-\,\omega(\mathbf{k}_3)]\, A^*(\mathbf{k}_1)\, A(\mathbf{k}_2)\, A(\mathbf{k}_3)\, d\mathbf{k}_1\, d\mathbf{k}_2\, d\mathbf{k}_3\,. \qquad (2.3)$$

Here

$$T(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)=$$
$$=-4\,\frac{\omega(\mathbf{k}_2+\mathbf{k}_3)\, V(-\mathbf{k}-\mathbf{k}_1, \mathbf{k}, \mathbf{k}_1)\, V(-\mathbf{k}_2-\mathbf{k}_3, \mathbf{k}_2, \mathbf{k}_3)}{\omega^2(\mathbf{k}_2+\mathbf{k}_3)-[\omega(\mathbf{k}_2)+\omega(\mathbf{k}_3)]^2}-$$
$$-4\,\frac{\omega(\mathbf{k}_1-\mathbf{k}_3)\, V(-\mathbf{k}, \mathbf{k}_2, \mathbf{k}-\mathbf{k}_2)\, V(-\mathbf{k}_3, \mathbf{k}_1, \mathbf{k}_3-\mathbf{k}_1)}{\omega^2(\mathbf{k}_3-\mathbf{k}_1)-[\omega(\mathbf{k}_3)-\omega(\mathbf{k}_1)]^2}-$$
$$-4\,\frac{\omega(\mathbf{k}_1-\mathbf{k}_2)\, V(-\mathbf{k}, \mathbf{k}_3, \mathbf{k}-\mathbf{k}_3)\, V(-\mathbf{k}_2, \mathbf{k}_1, \mathbf{k}_2-\mathbf{k}_1)}{\omega^2(\mathbf{k}_2-\mathbf{k}_1)-[\omega(\mathbf{k}_1)-\omega(\mathbf{k}_2)]^2}+$$
$$+W(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)\,. \qquad (2.4)$$

Obviously, the terms omitted from the Hamiltonian (1.11) cannot contribute to (2.4).

In using (2.3), we have to assume that $f \ll A$. For this condition to be satisfied, it is necessary that the denominators in (2.2) and (2.4) do not vanish. There is a zero denominator if

$$\omega(\mathbf{k})=\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2),\quad \mathbf{k}=\mathbf{k}_1+\mathbf{k}_2 \qquad (2.5)$$

have a solution.

If this system has no solutions, Eq. (2.4) can be used for sufficiently small a/λ, but if it has a solution we have to assume additional restrictions.

If $\omega(\mathbf{k})$ is a monotonic function, we note that a sufficient condition for the existence of a solution to (2.5) is

$$\omega(\mathbf{k})>\omega(\mathbf{k}_1)+\omega(\mathbf{k}-\mathbf{k}_1), \qquad (2.6)$$

where k and k_1 are in the direction of the same straight line. Indeed, if (2.6) holds, by adding to k_1 components perpendicular to k, we can increase the right-hand side of (2.6) and convert (2.6) into an equation. On the other hand, if the inequality converse to (2.6) holds, this is a sufficient condition for the absence of solutions to (2.5). For gravitational waves, with the dispersion law

$$\omega(\mathbf{k})=\sqrt{g\,|\mathbf{k}|}\,,$$

an inequality converse to (2.6) holds. Accordingly, (2.5) cannot have any solutions and for small a/λ (2.3) applies. For capillary waves $[k \gg (g/\alpha)^{1/2}]$, with dispersion law $\omega(\mathbf{k}) = \sqrt{\alpha\,|\,\mathbf{k}\,|^3}$, (2.6) holds, so that (2.3), in general, cannot be used; if it is assumed that the wave packet is sufficiently narrow, i.e., if $a(\mathbf{k})$ is nonzero for $|\,\mathbf{k}-\mathbf{k}_0\,| \ll \mathbf{k}_0$, Eq. (2.5) cannot be satisfied for any $\omega(\mathbf{k})$. Hence, assuming the wave packet is narrow, Eq. (2.6) is applicable for any dispersion laws, in particular, for capillary waves.

Assuming the wave packet is narrow, we can make a further simplification in Eq. (2.3). We introduce the variable $\varkappa = \mathbf{k}-\mathbf{k}_0$ and expand $\omega(\mathbf{k})$ in powers of $\varkappa$ to terms of second order:

$$\omega(\mathbf{k})=\omega(\mathbf{k}_0)+\varkappa_x C+{}^1\!/_2\,(\lambda_\parallel \varkappa_x^2+\lambda_\perp \varkappa_y^2)\,,$$
$$c=\frac{\partial\omega}{\partial k}\Big|_{k=k_0},\quad \lambda_\parallel=\frac{\partial^2\omega}{\partial k^2}\Big|_{k=k_0},\quad \lambda_\perp=\frac{c}{k_0}\,,\quad D_{\alpha\beta}=\frac{\partial^2\omega}{\partial k_\alpha k_\beta}\Big|_{k=k_0}.$$

Here $\varkappa_x$ and $\varkappa_y$ are the projections of the vector $\varkappa$ both parallel and perpendicular to the vector $\mathbf{k}_0$; c is the group velocity of the waves; $\lambda_\perp$ is an eigenvector of the tensor $D_{\alpha\beta}$. Next we replace the approximation $T(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ by $w = T(\mathbf{k}_0, \mathbf{k}_0, \mathbf{k}_0, \mathbf{k}_0)$ and introduce the variable

$$b(\mathbf{k})=A(\mathbf{k})\exp i\,[\varkappa_x c+{}^1\!/_2\,(\lambda_\parallel\varkappa_x^2+\lambda_\perp\varkappa_y^2)]\,t\,,$$
$$\frac{\partial b}{\partial t}+i\,(\varkappa_x c+{}^1\!/_2\lambda_\parallel\varkappa_x^2+{}^1\!/_2\lambda_\perp\varkappa_y^2)\, b=$$
$$=-iw\int b^*(\mathbf{k}_1)\, b(\mathbf{k}_2)\, b(\mathbf{k}_3)\, \delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\, d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3\,. \quad (2.7)$$

We note that $\lambda_\perp$ is always positive, while $\lambda_\parallel$ vanishes for $\mathbf{k}_0 = \mathbf{k}_0^*$:

$$k_0^*=\Big(\frac{2}{\sqrt{3}}-1\Big)^{1/2}\Big(\frac{g}{\alpha}\Big)^{1/2}\sim 0.4\Big(\frac{g}{\alpha}\Big)^{1/2}.$$

For $\mathbf{k}_0 < \mathbf{k}_0^*$, $\lambda_\parallel$ is negative, and for $\mathbf{k}_0 > \mathbf{k}_0^*$, $\lambda_\parallel$ is positive. We apply the inverse Fourier transformation with respect to $\varkappa$:

$$b(x, y, t)=\frac{1}{2\pi}\int b(\varkappa_x, \varkappa_y, t)\exp[-i\,(x\varkappa_x+y\varkappa_y)]\, d\varkappa_x d\varkappa_y\,.$$

Here b(x, y, t) is the envelope of the wave packet. We obtain

$$\frac{\partial b}{\partial t}+c\,\frac{\partial b}{\partial x}-\frac{i}{2}\Big(\lambda_\parallel\,\frac{\partial^2 b}{\partial x^2}+\lambda_\perp\,\frac{\partial^2 b}{\partial y^2}\Big)=-iw\,|\,b\,|^2 b\,. \qquad (2.8)$$

To simplify the equation further we introduce the variable $\xi = x - ct$ (which corresponds to transformation to a system of coordinates moving with a velocity equal to the group velocity of the wave); we assume that the solution depends only on t and $z = \xi\cos\alpha + y\sin\alpha$; we obtain

$$\frac{\partial\Psi}{\partial t}-\frac{i\lambda}{2}\,\frac{\partial^2\Psi}{\partial z^2}=-iw\,|\,\Psi\,|^2\,\Psi,$$
$$\lambda=\lambda_\parallel\cos^2\alpha+\lambda_\perp\sin^2\alpha\,. \qquad (2.9)$$

Equation (2.3) has the exact solution

$$A(\mathbf{k})=b_0\,\delta(\mathbf{k}-\mathbf{k}_0)\exp[-it\Omega(\mathbf{k})],\qquad \Omega(\mathbf{k})=w\,|\,b_0\,|^2. \ (2.10)$$

Here b_0 is an arbitrary constant. In terms of the variables η and Ψ, solution (2.10) has the form

$$\eta=a\cos(kx-\omega t),\ \ \Psi=a\sin(kx-\omega t),$$
$$a=\frac{k^{1/2}}{\pi\sqrt{2}\,\omega^{1/2}(k)}\,|\,b_0\,|\,,\quad \omega=\omega(k)+\Omega(k)\,.$$

Calculation yields

$$\Omega(\mathbf{k}) = |b_0|^2\left[\frac{4V(-2\mathbf{k}, \mathbf{k}, \mathbf{k})^2}{4\omega^2(\mathbf{k}) - \omega^2(2\mathbf{k})} - W(\mathbf{k}, \mathbf{k}, \mathbf{k}, \mathbf{k})\right] =$$

$$= \omega(\mathbf{k})(ka)^2\left[\frac{1}{2}\,\frac{\omega^2(\mathbf{k})}{4\omega^2(\mathbf{k}) - \omega^2(2\mathbf{k})} + \frac{1}{4} - \frac{3\alpha|\mathbf{k}|^3}{16\omega(\mathbf{k})}\right]. \quad (2.11)$$

In the limit, for small k this takes the form $\Omega(k) = {}^1/_2(ka)^2\omega(k)$, which coincides with the expression obtained by Stokes in 1847. Thus, solution (2.9) approximates a periodic wave of finite amplitude.

When

$$k = \left(\frac{\alpha}{2g}\right)^{1/2},$$

the frequency displacement becomes infinite; for large k, it is negative.

In the limit, as $k \to \infty$ we have

$$\Omega(k) = -\frac{1}{16}(ka)^2\,\omega(k).$$

3. Stability of steady waves of finite amplitude. We consider the development of small perturbations against the background of a steady periodic wave. We seek $a(\mathbf{k})$ in the form

$$a(\mathbf{k}) = b_0\delta(\mathbf{k} - \mathbf{k}_0)\,e^{-i\omega t} + \alpha(\mathbf{k}, t)\,e^{-i\omega t}, \quad \omega = \omega(\mathbf{k}_0) + \Omega(\mathbf{k}_0). \quad (3.1)$$

We assume $\alpha(\mathbf{k})$ to be small in the sense that

$$\int |\alpha(\mathbf{k})|\,d\mathbf{k} \ll |b_0|.$$

Now we make (1.15) linear in $\alpha(\mathbf{k})$. To do this, we consider only terms on the right-hand side which vary slowly with time. We obtain

$$\frac{\partial\alpha(\mathbf{k})}{\partial t} = -2ib_0e^{i\gamma t}V(-\mathbf{k}_0, \mathbf{k}_0, \mathbf{k}_0 - \mathbf{k})\,\alpha^*(\mathbf{k}_0 - \mathbf{k});$$
$$\gamma = \omega(\mathbf{k}) + \omega(\mathbf{k} - \mathbf{k}_0) - \omega(\mathbf{k}_0) - \Omega(\mathbf{k}_0). \quad (3.2)$$

Eliminating $\alpha^*(\mathbf{k}_0 - \mathbf{k})$ from (3.2), we obtain

$$\frac{\partial}{\partial t}\left(e^{-i\gamma t}\frac{\partial\alpha(\mathbf{k})}{\partial t}\right) =$$

$$= \frac{8\pi\omega(\mathbf{k}_0)}{|\mathbf{k}_0|}\,e^{-i\gamma t}a^2V(-\mathbf{k}_0, \mathbf{k}, \mathbf{k}_0 - \mathbf{k})\,\alpha(\mathbf{k}). \quad (3.3)$$

Equation (3.3) has a solution of the form

$$\alpha(\mathbf{k}) = ce^{qt},$$

$$q = {}^1/_2 i\gamma \pm \sqrt{|b_0|^2U^2(-\mathbf{k}_0, \mathbf{k}, \mathbf{k}_0 - \mathbf{k}) - {}^1/_4\gamma^2}. \quad (3.4)$$

Instability will occur if the expression under the square-root sign is positive. In order that there should be instability for arbitrarily small b_0, the equation $\gamma = 0$ should have a solution. If we neglect the small term $\Omega(\mathbf{k}_0)$ in this equation, we arrive at the system of equations (2.5). As was established in section 2, this system can be solved for capillary waves; thus, instability of this type occurs for capillary waves. Unstable wave vectors are concentrated near the surface $\omega(\mathbf{k}) = \omega(\mathbf{k}_1) + \omega(\mathbf{k} - \mathbf{k}_1)$ in a layer of thickness proportional to the amplitude. The maximum increment in the instability is of order $\mathrm{Re}\,q \sim (ka)\,\omega(\mathbf{k})$.

This type of instability is impossible for gravitational waves. However, for these waves slower instabilities are possible. We use Eq. (2.3) and substitute into it $A(\mathbf{k})$ in the form

$$A(\mathbf{k}) = b_0\delta(\mathbf{k} - \mathbf{k}_0)\,e^{-i\Omega(\mathbf{k}_0)t} + \alpha(\mathbf{k}, t).$$

If we linearize in $\alpha(\mathbf{k}, t)$, we obtain

$$\frac{\partial\alpha(\mathbf{k})}{\partial t} = 2T(\mathbf{k}, \mathbf{k}_0, \mathbf{k}_0, \mathbf{k})\,|b_0|^2\alpha(\mathbf{k}) +$$

$$+ e^{-2i\Omega(\mathbf{k}_0)t}\,T(\mathbf{k}, 2\mathbf{k}_0 - \mathbf{k}, \mathbf{k}_0, \mathbf{k}_0)\,b_0^2\alpha^*(2\mathbf{k}_0 - \mathbf{k}).$$

This equation can be reduced to an equation of form (3.2); it has a solution proportional to e^{qt}, where, for q, we have

$$q = {}^1/_2 i\delta \pm \sqrt{|b_0|^2T^2(\mathbf{k}, 2\mathbf{k}_0 - \mathbf{k}, \mathbf{k}_0, \mathbf{k}_0)^2 - {}^1/_4\delta^2},$$

$$\delta = \omega(\mathbf{k}) + \omega(2\mathbf{k}_0 - \mathbf{k}) - 2\omega(\mathbf{k}_0) + 2|b_0|^2[T(\mathbf{k}, \mathbf{k}_0, \mathbf{k}_0, \mathbf{k}) +$$
$$+ T(2\mathbf{k}_0 - \mathbf{k}, \mathbf{k}_0, \mathbf{k}_0, 2\mathbf{k}_0 - \mathbf{k}) - T(\mathbf{k}_0, \mathbf{k}_0, \mathbf{k}_0, \mathbf{k}_0)]. \quad (3.5)$$

Consider first the case

$$|\mathbf{k} - \mathbf{k}_0|\,\frac{d\omega}{d\mathbf{k}} \gg \omega\,|b|^2. \quad (3.6)$$

Then terms proportional to b^2 can be dropped from (3.5). The condition for the existence of instability for arbitrarily small amplitudes is $\delta = 0$, which is equivalent to the existence of solutions for the equations

$$2\omega(\mathbf{k}) = \omega(\mathbf{k}_1) + \omega(\mathbf{k}_2), \qquad 2\mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2. \quad (3.7)$$

Obviously, these equations have solutions if

$$\omega\left(\frac{\mathbf{k}_1 + \mathbf{k}_2}{2}\right) > \frac{\omega(\mathbf{k}_1) + \omega(\mathbf{k}_2)}{2} \quad (3.8)$$

(a sufficient condition), where the vectors $\mathbf{k}_1$ and $\mathbf{k}_2$ are parallel to the same straight line. Inequality (3.8) is the requirement that $\omega(\mathbf{k})$ be convex upwards. For gravitational waves,

$$\left[|\mathbf{k}| \ll \left(\frac{g}{\alpha}\right)^{1/2}\right]$$

and the inequality necessarily holds.

Conversely, for capillary waves, the inverse inequality holds, indicating that instability of this type is impossible.

Equation (3.7) defines a surface in k-space. Unstable vectors lie near this surface in a layer of thickness proportional to b^2. The maximum increment in the instability for gravitational waves is of order

$$\gamma \sim (ka)^2\,\omega(k).$$

There are higher-order instabilities corresponding to conservation laws for m:

$$n\omega(\mathbf{k}) = \omega(\mathbf{k}_1) + \omega(\mathbf{k}_2), \qquad n\mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2.$$

The order of the increment for such instabilities is $\gamma(k) \sim (ka)^n\omega(k)$.

All these instabilities can be called destructive.

We turn now to instabilities for which $|\mathbf{k}_1 - \mathbf{k}_0| \ll \mathbf{k}_0$. To study these we use Eq. (3.3) directly. The solution

$$b = b_0\exp(-iw|b_0|^2t)$$

corresponds to a completely finite amplitude.

We now seek the solution in the form

$$\Psi = \exp(-iw|b_0|^2t)\,\{b_0 + \alpha e^{-i\omega t + i\varkappa_0 z} + \alpha^* e^{i\omega t - i\varkappa_0 z}\},$$

$$\varkappa_0 \ll \mathbf{k}_0.$$

Then for ω we have

$$\omega = \pm\sqrt{\omega|b_0|^2\lambda\varkappa_0^2 + {}^1/_4\lambda^2\varkappa_0^4}. \quad (3.9)$$

We see from (3.9) that instability is possible if $w\lambda < 0$, instabilities only being excited for sufficiently small wave vectors

$$\varkappa_0^2 < \frac{4w}{\lambda}|b_0|^2.$$

We consider the case of different wave numbers for the surface waves.

1. In the region of wave numbers

$$k_0 < \sqrt{\sqrt{{}^3/_4} - 1}\,(g/\alpha)^{1/2},$$

where $w > 0$, $\lambda_\perp > 0$, and $\lambda_\| < 0$, the domain of instability in the plane $\varkappa_x$, $\varkappa_y$ is bounded by the inequalities $0 < |\lambda_\|\,\varkappa_x^2 - \lambda_\perp\varkappa_y^2 < 4|b|^2w$; i.e., it lies between the hyperbola

$$4|b|^2w = \lambda_\|\varkappa_x^2 - \lambda_\perp\varkappa_y^2$$

and its asymptotes.

2. In the domain

$$\sqrt{\sqrt{{}^4/_3} - 1}\,(g/\alpha)^{1/2} < k_0 < 1/\sqrt{2}\,(g/\alpha)^{1/2},$$

where $\lambda_\perp > 0$, $\lambda_\| > 0$, and $w > 0$, instability, in general, is impossible.

3. In the domain of capillary waves

$$k_0 > 1 / \sqrt{2}\,(g/\alpha)^{1/2}, \qquad \lambda_\perp > 0,\ \lambda_\parallel > 0,\ w < 0\,,$$

the region of instability is the interior of the ellipse

$$\lambda_\parallel \varkappa_x^2 + \lambda_\perp \varkappa_y^2 = 4|b|^2 w\,.$$

In (2.9) we make the change of variables

$$\Psi = \sqrt{n}\exp\left[\frac{i}{\lambda}\int^z v\,dz\right].$$

Equation (2.9) becomes

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial z}(nv) = 0,$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial z} = -w\lambda\frac{\partial n}{\partial z} + \frac{\lambda^2}{2}\frac{\partial}{\partial z}\frac{1}{\sqrt{n}}\frac{\partial^2}{\partial z^2}\sqrt{n}\,. \qquad (3.10)$$

These equations are similar to the equations of gasdynamics with an adiabatic relationship between the pressure and the density,

$$P = \frac{w\lambda n^2}{2}\,,$$

and differ from them by an additional term containing the third derivative with respect to z. If we consider a sufficiently large-scale motion with characteristic scale L, then for

$$\frac{1}{L} \ll \frac{2wn_0}{\lambda},$$

this term may be neglected. For positive pressure $w\lambda > 0$, Eq. (3.10) describes sound waves of velocity $\sqrt{w\lambda n_0}$. For negative pressure, the speed of sound becomes imaginary, which means that the initial perturbations increase exponentially as

$$v \sim \exp\left(t\sqrt{|w\lambda|\,n_0}\right).$$

Hence we have the case of a negative-pressure type of instability.

We note that (3.9) can be obtained for the increment in the negative pressure instability if we let $k \to k_0$ in (3.5). Thus, negative-pressure instability is the limiting case of slow destructive instability of gravitational waves.

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